Mathematics 1c, Spring 2008. Practice Midterm Examination

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Print Your Name:
Your Section:

- This exam has five questions.
- This exam should take about 3.5 to 4 hours to complete. The real midterm will be constructed so that students who are well versed in the material will require 3 hours. That exam will be a 3 hour exam. There is no credit for overtime work.
- No aids (including notes, books, calculators etc.) are permitted.
- The real exam must be turned in by ***noon on Wednesday, May 7.***
- All 5 questions should be answered on this exam, using the backs of the sheets or appended pages if necessary.
- Show all your work and justify all claims using plain and proper English.
- Each question is worth 20 points.
- **Good Luck !!**

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1. Consider the matrix
$$A = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

   (a) Is A diagonalizable?

   (b) Is A invertible?

   (c) Find the eigenvalues and eigenvectors of A

   (d) Compute $(A^{-1})^6$

   Solution.

   (a) Yes, because A is symmetric

   (b) Yes, because its determinant is -2, which is not zero

   (c) The characteristic polynomial is
$$p(x) = \det \begin{bmatrix} x - \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & x - \frac{1}{2} \end{bmatrix} = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$$
whose roots are $2, -1$. Thus, the eigenvalues are $2, -1$. Since A is symmetric, we know that the diagonalizing matrix can be chosen to be orthogonal. We verify this by computing the associated normalized eigenvectors to be $(1, -1)/\sqrt{2}$ and $(1, 1)/\sqrt{2}$.

   (d) Let Q be the diagonalizing matrix. It is given explicitly by
$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
Thus,
$$A = QDQ^{-1}$$
where $D = \mathrm{diag}(2, -1)$. Then
$$\begin{aligned} (A^{-1})^6 &= \left((QDQ^{-1})^{-1}\right)^6 = \left(QD^{-1}Q^{-1}\right)^6 \\ &= Q(D^{-1})^6 Q^{-1} = Q \begin{bmatrix} \frac{1}{2^6} & 0 \\ 0 & 1 \end{bmatrix} Q^{-1} = Q \begin{bmatrix} \frac{1}{2^6} & 0 \\ 0 & 1 \end{bmatrix} Q^T \end{aligned}$$
Multipying this out, we get
$$(A^{-1})^6 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2^7} & \frac{1}{2} - \frac{1}{2^7} \\ \frac{1}{2} - \frac{1}{2^7} & \frac{1}{2} + \frac{1}{2^7} \end{bmatrix}$$

There is no need to work this out numerically, but if you do, you get

$$(A^{-1})^6 = \begin{bmatrix} \frac{65}{128} & \frac{63}{128} \\ \frac{63}{128} & \frac{65}{128} \end{bmatrix}$$

2. Let A be a 3×3 orthogonal matrix.

   (a) Show that the determinant of A is ± 1.

   (b) Let λ be a real eigenvalue of A. Show that $\lambda = \pm 1$

   (c) Suppose that A is orthogonal and $\det A = 1$. Show that at least one eigenvalue of A equals 1. Must one of them equal -1 as well?

   (d) Give an example of a 3×3 orthogonal matrix that is not diagonalizable (as a real matrix).

   Solution.

   (a) Since $AA^T =$ Identity, taking determinants and using the fact that $\det A = \det A^T$ and that $\det \text{Identity} = 1$, we see that $(\det A)^2 = 1$ and so $\det A = \pm 1$.

   (b) If $Av = \lambda v$ then taking the inner product of both sides with $Av = \lambda v$, and using the fact that $\langle Av, Av \rangle = \left\langle A^T Av, v \right\rangle = \|v\|^2$, we get $\|v\|^2 = \lambda^2 \|v\|^2$ and so $\lambda^2 = 1$. Because λ is real, we get $\lambda = \pm 1$.

   (c) Since any cubic equation has at least one real root, we see that there is at least one root of the characteristic polynomial that is ± 1. If it equals 1 we are done. Suppose, alternatively, that it equals -1. Since $\det A = 1$ and the determinant is the product of the eigenvalues, the product of the other two eigenvalues must be -1. These other two eigenvalues are either real or have the form λ and $\overline{\lambda}$. But they cannot have the form λ and $\overline{\lambda}$ since the product of these two complex numbers is positive. Thus, these two other eigenvalues must be real. Thus, they are either plus or minus 1. But since their product is -1, at least one eigenvalue is one.

   The identity matrix shows that there need not be an eigenvalue equal to -1.

(d) There are many answers for this; a simple one is a rotation through $\pi/2$ around the z-axis, namely

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Do each of the following calculations

(a) Find the tangent vector to the curve

$$\boldsymbol{\sigma}(t) = e^{1-t}\mathbf{i} - t^2\mathbf{j} + \sin(\pi t/2)\mathbf{k}$$

at the point $t = 1$.

(b) If a particle following the curve in (a) flies off on a tangent at $t = 1$, where is it at $t = 2$?

(c) Find the gradient of the function

$$f(x, y, z) = yz\sin(\pi x) - xyz$$

at the point $(1, 1, 2)$.

(d) Find the equation of the tangent plane to the graph of

$$z = 3y^2 - x^3 + 2$$

at the point on the graph with $x = 1$ and $y = -1$.

(e) Calculate the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = -3r^{-2}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Verify that the gradient is orthogonal to the level sets of f.

Solution.

(a) The tangent vector to the curve is obtained by differentiating the curve:

$$\boldsymbol{\sigma}'(t) = -e^{1-t}\mathbf{i} - 2t\mathbf{j} + \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\mathbf{k}.$$

Evaluating this at $t = 1$, we get the required tangent vector $\boldsymbol{\sigma}'(1) = -\mathbf{i} - 2\mathbf{j}$.

(b) The equation for the particle path following the tangent line is

$$\ell(t) = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + (t-1)(-\mathbf{i} - 2\mathbf{j}).$$

Note that this line passes through the point $\boldsymbol{\sigma}(1)$ at $t = 1$ and has the direction $\boldsymbol{\sigma}'(1)$. Setting $t = 2$, we get the desired position to be $\ell(2) = (0, -3, 1)$.

(c) The gradient at a general point is:

$$\begin{aligned}\nabla f(x, y, z) = {} & (\pi yz\cos(\pi z) - yz)\mathbf{i} + (z\sin(\pi x) - xz)\mathbf{j} + \\ & + (y\sin(\pi x) - xy)\mathbf{k}\end{aligned}$$

Evaluating this at the required point, we get

$$\nabla f(1,\ 1,\ 2) = -2(\pi + 1)\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

(d) Denote

$$f(x, y) = 3y^2 - x^3 + 2; \quad \text{and} \quad (x_0, y_0) = (1,\ -1)$$

We have

$$\begin{aligned} f(x_0, y_0) &= 4 \\ f_x(x_0, y_0) &= -3x_0^2 = -3 \\ f_y(x_0, y_0) &= 6y_0 = -6 \end{aligned}$$

The equation of the tangent plane to the graph of $z = f(x,\ y)$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In our case, this becomes $z = 4 - 3(x-1) - 6(y+1)$ that is

$$z = -3x - 6y + 1.$$

(e) The function is

$$f(x, y, z) = \frac{-3}{x^2 + y^2 + z^2}$$

and so the gradient is

$$\begin{aligned}\nabla f(x, y, z) &= f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} = \\ &= \frac{6x}{(x^2 + y^2 + z^2)^2}\,\mathbf{i} + \frac{6y}{x^2 + y^2 + z^2)^2}\,\mathbf{j} + \frac{6z}{(x^2 + y^2 + z^2)^2}\,\mathbf{k} \\ &= \frac{6}{(x^2 + y^2 + z^2)^2}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= \frac{6\mathbf{r}}{r^4}\end{aligned}$$

4. Let $f(x, y, z)$ be a given (smooth) function defined on the whole of $\mathbb{R}^3$.

   (a) Let $\boldsymbol{\sigma}(t) = t\,\mathbf{i} - e^t\,\mathbf{j} - t^3\,\mathbf{k}$. Write the chain rule for $f \circ \boldsymbol{\sigma}$ in gradient and component notation.

   (b) If the gradient ∇f has a positive z component in the half space $z \geq 0$, must $f(1, 2, 1)$ be larger than $f(1, 2, 0)$? Must it be larger than $f(0, -1, 0)$? Prove or find a counter example.

   (c) If $f(x, y, z) = x^2 - y^2 + z^4$, find the derivative of f in the direction of the vector $\mathbf{i} - \mathbf{j}$ at the point $(1, -1, 1)$.

   (d) In what direction is the function in (c) increasing the fastest at $(1, -1, 1)$?

   (e) In what directions is the function in (c) increasing at half its maximum rate at the point $(1, -1, 1)$?

Solution.

(a) By the chain rule,

$$\begin{aligned}\frac{d}{dt} f \circ \boldsymbol{\sigma}(t) &= \nabla f(\boldsymbol{\sigma}(t)) \cdot \dot{\boldsymbol{\sigma}}(t) = \nabla f(\boldsymbol{\sigma}(t)) \cdot \big(\mathbf{i} - e^t\mathbf{j} - 3t^2\mathbf{k}\big) \\ &= \frac{\partial f}{\partial x}\mathbf{i} - e^t\frac{\partial f}{\partial y} - 3t^2\frac{\partial f}{\partial z}\mathbf{k}\end{aligned}$$

(b) The answer to the first part is yes; in fact, the rate of change of f along the *vertical line* from $(1, 2, 0)$ to $(1, 2, 1)$, namely $\mathbf{c}(t) = (1, 2, t)$ is $\nabla f(1, 2, t) \cdot \mathbf{k}$, where $0 \leq t \leq 1$, which is positive by assumption.

The answer to the second part is no; one possible counter example is (there are lots of counter examples) $f(x, y, z) = -2x + z$. Then $\nabla f = (-2, 0, 1)$ but $f(1, 2, 1) = -1 < 0 = f(0, -1, 0)$.

(c) Let $\mathbf{n} = \frac{1}{\sqrt{2}}(1, -1, 0)$ then the derivative of f in the direction $\mathbf{n}$ is given by

$$\nabla f \cdot \mathbf{n} = (2x, -2y, 4z^3) \cdot \frac{1}{\sqrt{2}}(1, -1, 0) = \frac{2}{\sqrt{2}}(x + y),$$

which equals zero at the point $(1, -1, 1)$.

(d) The function is increasing the fastest in the direction of the gradient, namely in the direction $\nabla f(1,-1,1) = (2,2,4)$. (You can normalize this vector if you like.)

(e) Let $\mathbf{n}$ be such a direction. Then the given condition reads

$$\nabla f(1,-1,1) \cdot \mathbf{n} = \frac{1}{2}\|\nabla f(1,-1,1)\|$$

since the direction of maximum rate of change is the direction of the gradient and the rate of change in that direction is

$$\nabla f(1,-1,1) \cdot \frac{\nabla f(1,-1,1)}{\|\nabla f(1,-1,1)\|} = \|\nabla f(1,-1,1)\|.$$

This means that $\cos\theta = \frac{1}{2}$ and so $\theta = 60°$, where θ is the angle between $\nabla f(1,-1,1) = (2,2,4)$ and $\mathbf{n}$. The collection of such vectors $\mathbf{n}$ form a cone around the axis $\nabla f(1,-1,1) = (2,2,4)$.

5. This problem has three main parts.

(a) Consider the two functions defined on $\mathbb{R}^3$ by

$$U = x^2 - y^2 + \sin z \quad \text{and} \quad V = xy\cos(xz).$$

i. Suppose that (x, y, z) are functions of new variables (u, v). Write out the chain rule for this situation, giving the derivatives of U and V as functions of (u, v) in matrix notation.

ii. Consider the specific situation in which $x = u - v$, $y = u + v$ and $z = u$. Calculate the derivative matrix of the resulting map of (u, v) to (U, V) evaluated at $u = 1$, $v = 0$.

(b) Find the extreme points of $f(x, y, z) = x + y + z$ subject to the *two constraints* $x^2 + y^2 = 5$ and $y + 2z = 3$.

(c) Let $f(x, y) = x^2 + 3xy + y^2 + 16$. Calculate the eigenvalues of the matrix of second partial derivatives of f at the origin. Using this information alone, determine if the origin is a maximum, a minimum or a saddle point.

Solution.

(a) Let F denote the mapping $(x, y, z) \mapsto (U, V)$ defined by these equations and let G denote the mapping $(u, v) \mapsto (x, y, z)$ (for (i) it is general, and will be made specific in (ii)).

i. The composition $F \circ G$ is the mapping $(u, v) \mapsto (U, V)$ obtained by replacing (x, y, z) by their expressions in terms of (u, v). The chain rule then says that

$$\mathbf{D}(F \circ G)(u, v) = \mathbf{D}F(x, y, z) \cdot \mathbf{D}G(u, v)$$

In matrix form, this equation reads as follows: follows:

$$\begin{bmatrix} \frac{\partial U}{\partial u} & \frac{\partial U}{\partial v} \\ \frac{\partial V}{\partial u} & \frac{\partial V}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & -2y & \cos z \\ y\cos(xz) - xyz\sin(xz) & x\cos(xz) & -x^2 y \sin(xz) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

ii. When $u = 1, v = 0$, we have $x = u - v = 1$, $y = u + v = 1$ and $z = u = 1$. Thus, the first matrix on the right hand side of the preceding equation becomes

$$\frac{\partial(U, V)}{\partial(x, y, z)}\Big|_{(1,1,1)} = \begin{bmatrix} 2 & -2 & \cos 1 \\ \cos 1 - \sin 1 & \cos 1 & -\sin 1 \end{bmatrix}.$$

Similarly, the second matrix on the right hand side is

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Multipying these matrices we get

$$\begin{bmatrix} \frac{\partial U}{\partial u} & \frac{\partial U}{\partial v} \\ \frac{\partial V}{\partial u} & \frac{\partial V}{\partial v} \end{bmatrix}\Bigg|_{(u,v)=(1,0)} = \begin{bmatrix} \cos 1 & -4 \\ 2\cos 1 - 2\sin 1 & \sin 1 \end{bmatrix}$$

(b) Write the two constraints as follows:

$$g_1(x, y, z) = x^2 + y^2 - 5 = 0$$

and

$$g_2(x, y, z) = y + 2z - 3 = 0.$$

The set of points satisfying both these constraints is (as was explained in lecture) a curve in $\mathbb{R}^3$. Thus, the maximum and minimum points p must be points where $\nabla f(p) = \lambda_1 \nabla g_1(p) + \lambda_2 \nabla g_2(p)$. Thus we must find x, y, z, λ_1, and λ_2 such that

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z)$$

and

$$\begin{aligned} g_1(x, y, z) &= 0, \\ g_2(x, y, z) &= 0. \end{aligned}$$

Computing the gradients and equating components, we get

$$\begin{aligned} 1 &= \lambda_1 \cdot 2x + \lambda_2 \cdot 0, \\ 1 &= \lambda_1 \cdot 2y + \lambda_2 \cdot 1, \\ 1 &= \lambda_1 \cdot 0 + \lambda_2 \cdot 2, \end{aligned}$$

and

$$x^2 + y^2 = 5,$$

$$y + 2z = 3.$$

These are five equations for x, y, z, λ_1, and λ_2. From the third, $\lambda_2 = 1/2$, and so $2x\lambda_1 = 1, 2y\lambda_1 = 1/2$. Since the first equation implies $\lambda_1 \neq 0$, we have $x = 2y$. Thus, from $x^2 + y^2 = 5$, we get $y = \pm 1$ and from $y + 2z = 3$ we get $z = 1, 2$. Hence the desired extrema are $(2, 1, 1)$ and $(-2, -1, 2)$. By inspection, $(2, 1, 1)$ gives a maximum, and $(-2, -1, 2)$ a minimum.

(c) Note that the origin is a critical point of f. The matrix of second derivatives is given by

$$S = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

The eigenvalues of this matrix are the roots of the characteristic polynomial, namely

$$p(\lambda) = \det \begin{bmatrix} \lambda - 2 & -3 \\ -3 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 9$$

which has roots $\lambda = 5, -1$.

Thus, after a rotation to new coordinates (X, Y) given by diagonalizing the second derivative matrix S, the quadratic function

$$f = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix} + 16$$

will take the form $f(X, Y) = \frac{1}{2}(5X^2 - Y^2) + 16$, so f clearly has a saddle point at the origin.

One can also see that f has a saddle point by direct inspection. Note that if $x = 0$, then $f(0, y) = y^2 + 16$ and similarly $f(x, 0) = x^2 + 16$. Thus, f is increasing along the x and y axes. However, in the direction $y = -x$, we have $f(x, -x) = -x^2 + 16$ and so f is decreasing along that direction.